

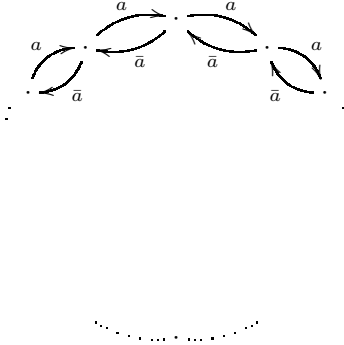
A FAMILY OF KOSZUL SELF-INJECTIVE ALGEBRAS WITH FINITE HOCHSCHILD COHOMOLOGY

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ABSTRACT. This paper presents an infinite family of Koszul self-injective algebras whose Hochschild cohomology ring is finite-dimensional. Moreover, for each $N \geq 5$ we give an example where the Hochschild cohomology ring has dimension N . This family of algebras includes and generalizes the 4-dimensional Koszul self-injective local algebras of [2], which were used to give a negative answer to Happel's question, in that they have infinite global dimension but finite-dimensional Hochschild cohomology.

INTRODUCTION

Let K be a field. Throughout this paper we suppose $m \geq 1$, and let \mathcal{Q} be the quiver with m vertices, labelled $0, 1, \dots, m-1$, and $2m$ arrows as follows:



Let a_i denote the arrow that goes from vertex i to vertex $i+1$, and let \bar{a}_i denote the arrow that goes from vertex $i+1$ to vertex i , for each $i = 0, \dots, m-1$ (with the obvious conventions modulo m). We denote the trivial path at the vertex i by e_i . Paths are written from left to right.

We define Λ to be the algebra $K\mathcal{Q}/I$ where I is the ideal of $K\mathcal{Q}$ generated by $a_i a_{i+1}$, $\bar{a}_{i-1} \bar{a}_{i-2}$ and $a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}$, for $i = 0, \dots, m-1$, where the subscripts are taken modulo m . These algebras are Koszul self-injective special biserial algebras and as such play an important role in various aspects of the representation theory of algebras. In particular, for m even, this algebra occurred in the presentation by quiver and relations of the Drinfeld double of the generalized Taft algebras studied in [4], and in the study of the representation theory of $U_q(\mathfrak{sl}_2)$, for which, see [3, 10, 14, 15].

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For $m \geq 1$ and for each $\mathbf{q} = (q_0, q_1, \dots, q_{m-1}) \in (K^*)^m$, we define $\Lambda_{\mathbf{q}} = K\mathcal{Q}/I_{\mathbf{q}}$, where $I_{\mathbf{q}}$ is the ideal of $K\mathcal{Q}$ generated by

$$a_i a_{i+1}, \bar{a}_{i-1} \bar{a}_{i-2}, q_i a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1} \text{ for } i = 0, \dots, m-1.$$

These algebras are socle deformations of the algebra L , with $\Lambda_{\mathbf{q}} = L$ when $\mathbf{q} = (1, 1, \dots, 1)$, and were studied in [13]. We are assuming each q_i is non-zero since we wish to study self-injective algebras. Indeed, the algebra $\Lambda_{\mathbf{q}}$ is a Koszul self-injective socle deformation of L , and the K -dimension of $\Lambda_{\mathbf{q}}$ is $4m$.

In the case $m = 1$, the algebras $\Lambda_{\mathbf{q}}$ were studied in [2], where they were used to answer negatively a question of Happel, in that their Hochschild cohomology ring is finite-dimensional but they are of infinite global dimension when $q \in K^*$ is not a root of unity. In this paper we show, for all $m \geq 1$, that the algebras $\Lambda_{\mathbf{q}}$, where $\mathbf{q} = (q_0, q_1, \dots, q_{m-1}) \in (K^*)^m$, all have finite-dimensional Hochschild cohomology ring when $q_0 q_1 \cdots q_{m-1}$ is not a root of unity. Thus, for each non-zero element of K which is not a root of unity, we have generalized the 4-dimensional algebra of [2] to an infinite family of algebras which all give a negative answer to Happel's question. This also complements the paper of Bergh and Erdmann [1] in which they extended the example of [2] by producing a family of local algebras of infinite global dimension for which the Hochschild cohomology ring is finite-dimensional. We remark that the algebras of [1, 2] are local algebras with 5-dimensional Hochschild cohomology ring. In this paper we give, for each $N \geq 5$, a finite-dimensional algebra with $m = N - 4$ simple modules and of infinite global dimension whose Hochschild cohomology ring is N -dimensional.

For a finite-dimensional K -algebra \mathcal{A} with Jacobson radical \mathfrak{r} , the Hochschild cohomology ring of \mathcal{A} is given by $\mathrm{HH}^*(\mathcal{A}) = \mathrm{Ext}_{\mathcal{A}^e}^*(\mathcal{A}, \mathcal{A}) = \bigoplus_{n \geq 0} \mathrm{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, \mathcal{A})$ with the Yoneda product, where $\mathcal{A}^e = \mathcal{A}^{\mathrm{op}} \otimes_K \mathcal{A}$ is the enveloping algebra of \mathcal{A} . Since all tensors are over the field K we write \otimes for \otimes_K throughout. We denote by \mathcal{N} the ideal of $\mathrm{HH}^*(\mathcal{A})$ which is generated by all homogeneous nilpotent elements. Thus $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$ is a commutative K -algebra.

The Hochschild cohomology ring modulo nilpotence of $\Lambda_{\mathbf{q}}$, where $\mathbf{q} = (q_0, q_1, \dots, q_{m-1}) \in (K^*)^m$, was explicitly determined in [13], where it was shown that $\mathrm{HH}^*(\Lambda_{\mathbf{q}})/\mathcal{N}$ is a commutative finitely generated K -algebra of Krull dimension 2 when $q_0 \cdots q_{m-1}$ is a root of unity, and is K otherwise. Note that, by setting $\mathbf{z} = (q_0 q_1 \cdots q_{m-1}, 1, \dots, 1)$, we have an isomorphism $\Lambda_{\mathbf{q}} \cong L_{\mathbf{z}}$ induced by $a_i \mapsto q_0 q_1 \cdots q_i a_i, \bar{a}_i \mapsto \bar{a}_i$. However, for ease of notation, we will consider the algebra in the form $\Lambda_{\mathbf{q}} = K\mathcal{Q}/I_{\mathbf{q}}$ with $\mathbf{q} = (q_0, q_1, \dots, q_{m-1}) \in (K^*)^m$. It was shown by Erdmann and Solberg in [6, Proposition 2.1] that, if $q_0 q_1 \cdots q_{m-1}$ is a root of unity, then the finite generation condition **(Fg)** holds, so that $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ is a finitely generated noetherian K -algebra. (See [5, 6, 11] for more details on the finite generation condition **(Fg)** and the rich theory of support varieties for modules over algebras which satisfy this condition.)

The aim of this paper is to determine $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ for each $m \geq 1$ in the case where $q_0 q_1 \cdots q_{m-1}$ is not a root of unity, and in particular to show that this ring is finite-dimensional. Thus we set $\zeta = q_0 q_1 \cdots q_{m-1} \in K^*$ and assume that ζ is not a root of unity.

1. THE PROJECTIVE RESOLUTION OF $\Lambda_{\mathbf{q}}$

A minimal projective bimodule resolution for Λ was given in [12, Theorem 1.2]. Since $\Lambda_{\mathbf{q}}$ is a Koszul algebra, we again use the approach of [7] and [8] and modify the resolution for Λ from [12] to give a minimal projective bimodule resolution (P^*, ∂^*) for $\Lambda_{\mathbf{q}}$.

We recall from [9], that the multiplicity of $\Lambda_{\mathbf{q}} e_i \otimes e_j \Lambda_{\mathbf{q}}$ as a direct summand of P^n is equal to the dimension of $\mathrm{Ext}_{\Lambda_{\mathbf{q}}}^n(S_i, S_j)$, where S_i, S_j are the simple right $\Lambda_{\mathbf{q}}$ -modules corresponding to the vertices i, j respectively. Thus the projective bimodules P^n are the same as those in the minimal projective bimodule resolution for Λ , and we have, for $n \geq 0$, that

$$P^n = \bigoplus_{i=0}^{m-1} [\bigoplus_{r=0}^n \Lambda_{\mathbf{q}} e_i \otimes e_{i+n-2r} \Lambda_{\mathbf{q}}].$$

Write $\mathfrak{o}(\alpha)$ for the trivial path corresponding to the origin of the arrow α , so that $\mathfrak{o}(a_i) = e_i$ and $\mathfrak{o}(\bar{a}_i) = e_{i+1}$. We write $\mathfrak{t}(\alpha)$ for the trivial path corresponding to the terminus of the arrow α , so that $\mathfrak{t}(a_i) = e_{i+1}$ and $\mathfrak{t}(\bar{a}_i) = e_i$. Recall that a non-zero element $r \in K\mathcal{Q}$ is said to be uniform if there are vertices v, w such that $r = vr = rw$. We then write $v = \mathfrak{o}(r)$ and $w = \mathfrak{t}(r)$.

In [8], the authors give an explicit inductive construction of a minimal projective resolution of \mathcal{A}/\mathfrak{t} as a right \mathcal{A} -module, for a finite-dimensional K -algebra \mathcal{A} . For $\mathcal{A} = K\Gamma/I$ and finite-dimensional, they define g^0 to be the set of vertices of Γ , g^1 to be the set of arrows of Γ , and g^2 to be a minimal set of uniform relations in the generating set of I , and then show that there are subsets $g^n, n \geq 3$, of $K\Gamma$, where $x \in g^n$ are uniform elements satisfying $x = \sum_{y \in g^{n-1}} y r_y = \sum_{z \in g^{n-2}} z s_z$ for unique $r_y, s_z \in K\Gamma$, which can be chosen in such a way that there is a minimal projective \mathcal{A} -resolution of the form

$$\cdots \rightarrow Q^4 \rightarrow Q^3 \rightarrow Q^2 \rightarrow Q^1 \rightarrow Q^0 \rightarrow \mathcal{A}/\mathfrak{t} \rightarrow 0$$

having the following properties:

- (1) for each $n \geq 0$, $Q^n = \coprod_{x \in g^n} \mathfrak{t}(x)\mathcal{A}$,
- (2) for each $x \in g^n$, there are unique elements $r_j \in K\Gamma$ with $x = \sum_j g_j^{n-1} r_j$,
- (3) for each $n \geq 1$, using the decomposition of (2), for $x \in g^n$, the map $Q^n \rightarrow Q^{n-1}$ is given by

$$\mathfrak{t}(x)a \mapsto \sum_j r_j \mathfrak{t}(x)a \quad \text{for all } a \in \mathcal{A},$$

where the elements of the set g^n are labelled by $g^n = \{g_j^n\}$. Thus the maps in this minimal projective resolution of \mathcal{A}/\mathfrak{t} as a right \mathcal{A} -module are described by the elements r_j which are uniquely determined by (2).

For our algebra $\Lambda_{\mathbf{q}}$, we now define sets g^n in the path algebra $K\mathcal{Q}$ which we will use to label the generators of P^n .

Definition 1.1. For the algebra $\Lambda_{\mathbf{q}}$, $i = 0, 1, \dots, m-1$ and $r = 0, 1, \dots, n$, define

$$g_{0,i}^0 = e_i$$

and, inductively for $n \geq 1$,

$$g_{r,i}^n = g_{r,i}^{n-1} a_{i+n-2r-1} + (-1)^n q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r} g_{r-1,i}^{n-1} \bar{a}_{i+n-2r}$$

with the conventions that $g_{-1,i}^{n-1} = 0$ and $g_{n,i}^{n-1} = 0$ for all n, i , and that $q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r} = 1$ if $r = n$.

Define $g^n = \bigcup_{i=0}^{m-1} \{g_{r,i}^n \mid r = 0, \dots, n\}$.

It is easy to see, for $n = 1$, that $g_{0,i}^1 = a_i$ and $g_{1,i}^1 = -\bar{a}_{i-1}$, whilst, for $n = 2$, we have $g_{0,i}^2 = a_i a_{i+1}$, $g_{1,i}^2 = q_i a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}$ and $g_{2,i}^2 = -\bar{a}_{i-1} \bar{a}_{i-2}$. Thus

$$\begin{aligned} g^0 &= \{e_i \mid i = 0, \dots, m-1\}, \\ g^1 &= \{a_i, -\bar{a}_i \mid i = 0, \dots, m-1\}, \\ g^2 &= \{a_i a_{i+1}, \quad q_i a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}, \quad -\bar{a}_{i-1} \bar{a}_{i-2} \text{ for all } i\}, \end{aligned}$$

so that g^2 is a minimal set of uniform relations in the generating set of $I_{\mathbf{q}}$.

Moreover, $g_{r,i}^n \in e_i(K\mathcal{Q})e_{i+n-2r}$, for $i = 0, \dots, m-1$ and $r = 0, \dots, n$. Since the elements $g_{r,i}^n$ are uniform elements, we may define $\mathfrak{o}(g_{r,i}^n) = e_i$ and $\mathfrak{t}(g_{r,i}^n) = e_{i+n-2r}$. Then

$$P^n = \oplus_{i=0}^{m-1} [\oplus_{r=0}^n \Lambda_{\mathbf{q}} \mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n) \Lambda_{\mathbf{q}}].$$

To describe the map $\partial^n: P^n \rightarrow P^{n-1}$, we need the following lemma and some notation.

lemma:maps

Lemma 1.2. For the algebra $\Lambda_{\mathbf{q}}$, for $n \geq 1$, $i = 0, 1, \dots, m-1$ and $r = 0, 1, \dots, n$, we have:

$$\begin{aligned} g_{r,i}^n &= g_{r,i}^{n-1} a_{i+n-2r-1} + (-1)^n \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r}}_{n-r \text{ terms}} g_{r-1,i}^{n-1} \bar{a}_{i+n-2r} \\ &= (-1)^r \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_i}_{r \text{ terms}} a_i g_{r,i+1}^{n-1} + (-1)^r \bar{a}_{i-1} g_{r-1,i-1}^{n-1} \end{aligned}$$

with the conventions that $g_{-1,i}^n = 0$ and $g_{n,i}^{n-1} = 0$ for all n, i , and that $q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r} = 1$ if $r = n$ and $q_{i-r+1} q_{i-r+2} \cdots q_i = 1$ if $r = 0$. Thus

$$g_{0,i}^n = g_{0,i}^{n-1} a_{i+n-1} = a_i g_{0,i+1}^{n-1} \text{ and } g_{n,i}^n = (-1)^n g_{n-1,i}^{n-1} \bar{a}_{i-n} = (-1)^n \bar{a}_{i-1} g_{n-1,i-1}^{n-1}.$$

Proof. The first formula is of course the definition of $g_{r,i}^n$ so we need to prove the second equality. We prove this by induction on n . We note that, with the above conventions, the second formula is correct for $n = 1$ and 2.

Suppose the second formula is true for n and $n-1$; we consider the case with $n+1$ and look at the difference:

$$\begin{aligned}
& g_{r,i}^n a_{i+n-2r} + (-1)^{n+1} q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} g_{r-1,i}^n \bar{a}_{i+n-2r+1} \\
& - (-1)^r q_{i-r+1} q_{i-r+2} \cdots q_i a_i g_{r,i+1}^n - (-1)^r \bar{a}_{i-1} g_{r-1,i-1}^n \\
& = (-1)^r q_{i-r+1} q_{i-r+2} \cdots q_i a_i g_{r,i+1}^{n-1} a_{i+n-2r} + (-1)^r \bar{a}_{i-1} g_{r-1,i-1}^{n-1} a_{i+n-2r} \\
& + (-1)^{n+1} q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} (-1)^{r-1} q_{i-r+2} q_{i-r+3} \cdots q_i a_i g_{r-1,i+1}^{n-1} \bar{a}_{i+n-2r+1} \\
& + (-1)^{n+1} q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} (-1)^{r-1} \bar{a}_{i-1} g_{r-2,i-1}^{n-1} \bar{a}_{i+n-2r+1} \\
& - (-1)^r q_{i-r+1} q_{i-r+2} \cdots q_i a_i g_{r,i+1}^{n-1} a_{i+n-2r} \\
& - (-1)^r q_{i-r+1} q_{i-r+2} \cdots q_i a_i (-1)^n q_{i-r+2} q_{i-r+3} \cdots q_{i+n-2r+1} g_{r-1,i+1}^{n-1} \bar{a}_{i+n-2r+1} \\
& - (-1)^r \bar{a}_{i-1} g_{r-1,i-1}^{n-1} a_{i+n-2r} - (-1)^r \bar{a}_{i-1} (-1)^n q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} g_{r-2,i-1}^{n-1} \bar{a}_{i+n-2r+1} \\
& = 0
\end{aligned}$$

as required. \square

In order to define ∂^n for $n \geq 1$ in a minimal projective bimodule resolution (P^*, ∂^*) of $\Lambda_{\mathbf{q}}$, we use the following notation. In describing the image of $\mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n)$ under ∂^n in the projective module P^{n-1} , we use subscripts under \otimes to indicate the appropriate summands of the projective module P^{n-1} . Specifically, let \otimes_r denote a term in the summand of P^{n-1} corresponding to $g_{r,-}^{n-1}$, and \otimes_{r-1} denote a term in the summand of P^{n-1} corresponding to $g_{r-1,-}^{n-1}$, where the appropriate index $-$ of the vertex may always be uniquely determined from the context. Indeed, since the relations are uniform along the quiver, we can also take labelling elements defined by a formula independent of i , and hence we omit the index i when it is clear from the context. Recall that nonetheless all tensors are over K .

The algebra $\Lambda_{\mathbf{q}}$ is Koszul, so we now use [7] to give a minimal projective bimodule resolution (P^*, ∂^*) of $\Lambda_{\mathbf{q}}$. We define the map $\partial^0: P^0 \rightarrow \Lambda_{\mathbf{q}}$ to be the multiplication map. For $n \geq 1$, we define the map $\partial^n: P^n \rightarrow P^{n-1}$ as follows:

$$\begin{aligned}
\partial^n: \mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n) \mapsto & (e_i \otimes_r a_{i+n-2r-1} + (-1)^n \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r}}_{n-r \text{ terms}} e_i \otimes_{r-1} \bar{a}_{i+n-2r}) \\
& + (-1)^n ((-1)^r \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_i a_i}_{r \text{ terms}} \otimes_r e_{i+n-2r} + (-1)^r \bar{a}_{i-1} \otimes_{r-1} e_{i+n-2r}).
\end{aligned}$$

Using our conventions, the degenerate cases $r=0$ and $r=n$ simplify to

$$\partial^n: \mathfrak{o}(g_{0,i}^n) \otimes \mathfrak{t}(g_{0,i}^n) \mapsto e_i \otimes_0 a_{i+n-1} + (-1)^n a_i \otimes_0 e_{i+n}$$

where the first term is in the summand corresponding to $g_{0,i}^{n-1}$ and the second term is in the summand corresponding to $g_{0,i+1}^{n-1}$, whilst

$$\partial^n: \mathfrak{o}(g_{n,i}^n) \otimes \mathfrak{t}(g_{n,i}^n) \mapsto (-1)^n e_i \otimes_{n-1} \bar{a}_{i-n} + \bar{a}_{i-1} \otimes_{n-1} e_{i-n},$$

with the first term in the summand corresponding to $g_{n-1,i}^{n-1}$ and the second term in the summand corresponding to $g_{n-1,i-1}^{n-1}$.

We claim that the map ∂^n does indeed make (P^*, ∂^*) into a complex.

lem:complex

Lemma 1.3. *We have $\partial^n \circ \partial^{n+1} = 0$.*

Proof. The proof is a matter of applying the two different recursive formulae for $g_{r,i}^n$. It is not difficult, but care is needed with all the terms. We have

$$\begin{aligned}
& \partial^n \circ \partial^{n+1}(\mathfrak{o}(g_{r,i}^{n+1}) \otimes \mathfrak{t}(g_{r,i}^{n+1})) \\
&= \partial^n((e_i \otimes_r a_{i+n-2r} + (-1)^{n+1} q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} e_i \otimes_{r-1} \bar{a}_{i+n-2r+1}) \\
&\quad + (-1)^{n+1}((-1)^r q_{i-r+1} q_{i-r+2} \cdots q_i a_i \otimes_r e_{i+n-2r+1} + (-1)^r \bar{a}_{i-1} \otimes_{r-1} e_{i+n-2r+1})) \\
&= \partial^n((\mathfrak{o}(g_{r,i}^n) \otimes_r \mathfrak{t}(g_{r,i}^n) a_{i+n-2r} + (-1)^{n+1} q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} \mathfrak{o}(g_{r-1,i}^n) \otimes_{r-1} \mathfrak{t}(g_{r-1,i}^n) \bar{a}_{i+n-2r+1}) \\
&\quad - (-1)^n((-1)^r q_{i-r+1} q_{i-r+2} \cdots q_i a_i \mathfrak{o}(g_{r,i+1}^n) \otimes_r \mathfrak{t}(g_{r,i+1}^n) + (-1)^r \bar{a}_{i-1} \mathfrak{o}(g_{r-1,i-1}^n) \otimes_{r-1} \mathfrak{t}(g_{r-1,i-1}^n))) \\
&= e_i \otimes_r a_{i+n-2r-1} a_{i+n-2r} + (-1)^n q_{i-r+1} \cdots q_{i+n-2r} e_i \otimes_{r-1} \bar{a}_{i+n-2r} a_{i+n-2r} \\
&\quad + (-1)^{n+r} q_{i-r+1} \cdots q_i a_i \otimes_r a_{i+n-2r} + (-1)^{n+r} \bar{a}_{i+n-2r} \otimes_{r-1} a_{i+n-2r} \\
&\quad + (-1)^{n+1} q_{i-r+1} \cdots q_{i+n-2r+1} e_i \otimes_{r-1} a_{i+n-2r+1} \bar{a}_{i+n-2r+1} \\
&\quad + (-1)^{2n+1} q_{i-r+1} \cdots q_{i+n-2r+1} q_{i-r+2} \cdots q_{i+n-2r+2} e_i \otimes_{r-2} \bar{a}_{i+n-2r+2} \bar{a}_{i+n-2r+1} \\
&\quad + (-1)^{2n+r} q_{i-r+1} \cdots q_{i+n-2r+1} q_{i-r+2} \cdots q_i a_i \otimes_{r-1} \bar{a}_{i+n-2r+1} \\
&\quad + (-1)^{2n+r} q_{i-r+1} \cdots q_{i+n-2r+1} \bar{a}_{i-1} \otimes_{r-2} \bar{a}_{i+n-2r+1} \\
&\quad - (-1)^{n+r} q_{i-r+1} \cdots q_i a_i \otimes_r a_{i+n-2r} - (-1)^{2n+r} q_{i-r+1} \cdots q_i q_{i-r+2} \cdots q_{i+n-2r+1} a_i \otimes_{r-1} \bar{a}_{i+n-2r+1} \\
&\quad - (-1)^{2n+r} q_{i-r+1} \cdots q_i q_{i-r+2} \cdots q_{i+n-2r+1} a_i a_{i+1} \otimes_r e_{i+n-2r+1} \\
&\quad - (-1)^{2n+2r} q_{i-r+1} \cdots q_i a_i \bar{a}_i \otimes_{r-1} e_{i+n-2r+1} \\
&\quad - (-1)^{n+r} \bar{a}_{i-1} \otimes_{r-1} a_{i+n-2r} - (-1)^{2n+r} q_{i-r+1} \cdots q_{i+n-2r+1} \bar{a}_{i-1} \otimes_{r-2} \bar{a}_{i+n-2r+1} \\
&\quad - (-1)^{2n+2r-1} q_{i-r+1} \cdots q_{i-1} \bar{a}_{i-1} a_{i-1} \otimes_r e_{i+n-2r+1} - (-1)^{2n+2r} \bar{a}_{i-1} \bar{a}_{i-2} \otimes_{r-1} e_{i+n-2r+1}
\end{aligned}$$

The third term cancels with the 9th term, the 4th with the 13th, the 7th with the 10th and the 8th with the 14th. We now apply the relations in $\Lambda_{\mathbf{q}}$. Using $a_i a_{i+1} = 0 = \bar{a}_i \bar{a}_{i-1}$, we have that the first, 6th, 11th and 16th terms are zero. The $q_i a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}$ relations mean that the 2nd and 5th terms cancel, and the 12th and 15th terms cancel. Thus the net sum is zero, and the result follows. \square

The next theorem is now immediate from [7, Theorem 2.1].

Theorem 1.4. *With the above notation, (P^*, ∂^*) is a minimal projective bimodule resolution of $\Lambda_{\mathbf{q}}$.*

2. THE HOCHSCHILD COHOMOLOGY RING OF $\Lambda_{\mathbf{q}}$

We consider the complex $\text{Hom}_{\Lambda_{\mathbf{q}}^e}(P^n, \Lambda_{\mathbf{q}})$. All our homomorphisms are $\Lambda_{\mathbf{q}}^e$ -homomorphisms and so we write $\text{Hom}(-, -)$ for $\text{Hom}_{\Lambda_{\mathbf{q}}^e}(-, -)$. We start by computing the dimension of the space $\text{Hom}(P^n, \Lambda_{\mathbf{q}})$ for each $n \geq 0$. For $m \geq 3$, we write $n = pm + t$ where $p \geq 0$ and $0 \leq t \leq m - 1$.

lem:dim

Lemma 2.1. *Suppose $m \geq 3$ and $n = pm + t$ where $p \geq 0$ and $0 \leq t \leq m - 1$. Then*

$$\dim_K \text{Hom}(P^n, \Lambda_{\mathbf{q}}) = \begin{cases} (4p + 2)m & \text{if } t \neq m - 1 \\ (4p + 4)m & \text{if } t = m - 1. \end{cases}$$

If $m = 1$ or $m = 2$ then

$$\dim_K \text{Hom}(P^n, \Lambda_{\mathbf{q}}) = 4(n + 1).$$

The proof is as for the non-deformed case (with $q_0 = q_1 = \dots = q_{m-1} = 1$) in [12, Lemma 1.7] and where $N = 1$, and so is omitted.

Applying $\text{Hom}(-, \Lambda_{\mathbf{q}})$ to the resolution (P^*, ∂^*) gives the complex $(\text{Hom}(P^n, \Lambda_{\mathbf{q}}), d^n)$ where $d^n : \text{Hom}(P^n, \Lambda_{\mathbf{q}}) \rightarrow \text{Hom}(P^{n+1}, \Lambda_{\mathbf{q}})$ is induced by the map $\partial^{n+1} : P^{n+1} \rightarrow P^n$. The n th Hochschild cohomology group $\text{HH}^n(\Lambda_{\mathbf{q}})$ is then given by $\text{HH}^n(\Lambda_{\mathbf{q}}) = \text{Ker } d^n / \text{Im } d^{n-1}$. We start by calculating the dimensions of $\text{Ker } d^n$ and $\text{Im } d^{n-1}$. We consider the cases $m \geq 3$ and $m = 2$ separately, and recall that the Hochschild cohomology of $\Lambda_{\mathbf{q}}$ in the case $m = 1$ was fully determined in [2].

We keep to the notational conventions of [12]. So far, we have simplified notation by denoting the idempotent $\mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n)$ of the summand $\Lambda_{\mathbf{q}} \mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n) \Lambda_{\mathbf{q}}$ of P^n uniquely by $e_i \otimes_r e_{i+n-2r}$ where $0 \leq i \leq m - 1$. However, even this notation with subscripts under the tensor product symbol becomes cumbersome in computations. Thus we now recall the additional conventions of [12, 1.3] which we keep throughout the rest of the paper. Specifically, since $e_{i+n-2r} \in \{e_0, e_1, \dots, e_{m-1}\}$, it would be usual to reduce the subscript $i + n - 2r$ modulo m . However, to make it explicitly clear to which summand of the projective module P^n we are referring and thus to avoid confusion, whenever we write $e_i \otimes e_{i+k}$ for an element of P^n , we will always have $i \in \{0, 1, \dots, m - 1\}$ and consider $i + k$ as an element of \mathbb{Z} , in that $r = (n - k)/2$ and $e_i \otimes e_{i+k} = e_i \otimes_{\frac{n-k}{2}} e_{i+k}$ and thus lies in the $\frac{n-k}{2}$ -th summand of P^n . We do not reduce $i + k$ modulo m in any of our computations. In this way, when considering elements in P^n , our element $e_i \otimes e_{i+k}$ corresponds uniquely to the idempotent $\mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n)$ of P^n with $r = (n - k)/2$, for each $i = 0, 1, \dots, m - 1$.

With this notation and for future reference, we note that an element $f \in \text{Hom}(P^n, \Lambda_{\mathbf{q}})$ is determined by its image on each $e_i \otimes e_j$ that generates a summand of P^n . Now $f(e_i \otimes e_j) \in e_i \Lambda_{\mathbf{q}} e_j$ and hence can only be non-zero if $i = j$ or if $i = j \pm 1$. For $m \geq 3$ and $f \in \text{Hom}(P^n, \Lambda_{\mathbf{q}})$ we may

write:

$$\begin{cases} f(e_i \otimes e_{i+\alpha m}) = \sigma_i^\alpha e_i + \tau_i^\alpha \bar{a}_{i-1} a_{i-1}, \\ f(e_i \otimes e_{i+\beta m-1}) = \lambda_i^\beta \bar{a}_{i-1}, \\ f(e_i \otimes e_{i+\gamma m+1}) = \mu_i^\gamma a_i, \end{cases}$$

with coefficients $\sigma_i^\alpha, \tau_i^\alpha, \lambda_i^\beta$ and μ_i^γ in K , and appropriate ranges of integers α, β and γ . Specifically, for $\Lambda_{\mathbf{q}} e_i \otimes e_{i+\alpha m} \Lambda_{\mathbf{q}}$ to be a summand of P^n , we require $i + \alpha m = i + n - 2r$ for some $0 \leq r \leq n$. Similarly we require $i + \beta m - 1 = i + n - 2r$ and $i + \gamma m + 1 = i + n - 2r$ for some $0 \leq r \leq n$. The precise ranges of α, β and γ for the case $m \geq 3$ are as follows. (We have four cases based on the parity of t and of m , where $n = pm + t$ with $0 \leq t \leq m - 1$.)

If both t and m are even, then we only need α . We have $2p + 1$ values of α with $-p \leq \alpha \leq p$.

If t is even and m is odd, then we have $p + 1$ values of α with $-p \leq \alpha \leq p$ and $\alpha \equiv p \pmod{2}$. For $t \leq m - 2$ we also have p values of β and γ with $-p + 1 \leq \beta \leq p - 1$, $-p + 1 \leq \gamma \leq p - 1$ and $\beta \equiv \gamma \equiv p + 1 \pmod{2}$. If $t = m - 1$ then we get $p + 1$ values of β and γ with $-p + 1 \leq \beta \leq p + 1$, $-p - 1 \leq \gamma \leq p - 1$ and $\beta \equiv \gamma \equiv p + 1 \pmod{2}$.

If t is odd and m is even, then we have no values for α . For $t \leq m - 2$ we have $2p + 1$ values of β and γ with $-p \leq \beta \leq p$ and $-p \leq \gamma \leq p$. If $t = m - 1$ then we get $2p + 2$ values of β and γ with $-p \leq \beta \leq p + 1$ and $-p - 1 \leq \gamma \leq p$.

If t is odd and m is odd, then we have p values of α with $-p + 1 \leq \alpha \leq p - 1$ and $\alpha \equiv p + 1 \pmod{2}$. We also have $p + 1$ values of β and γ with $-p \leq \beta \leq p$, $-p \leq \gamma \leq p$ and $\beta \equiv \gamma \equiv p \pmod{2}$.

We consider the case $m = 2$ in Section 3 and suppose now that $m \geq 3$.

2.1. Ker d^n where $m \geq 3$. Let $f \in \text{Hom}(P^n, \Lambda_{\mathbf{q}})$ and suppose $f \in \text{Ker } d^n$ so that $d^n(f) = f \circ \partial^{n+1} \in \text{Hom}(P^{n+1}, \Lambda_{\mathbf{q}})$. Write $n = pm + t$ with $0 \leq t \leq m - 1$. We evaluate $d^n(f)$ at $e_i \otimes e_{i+n+1-2r}$ for $r = 0, \dots, n + 1$. We have three separate cases for r to consider.

We first consider $r = 0$. Then, for each $i = 0, \dots, m - 1$ we have

$$\begin{aligned} d^n(f)(e_i \otimes e_{i+n+1}) &= f \circ \partial^{n+1}(e_i \otimes e_{i+n+1}) \\ &= f(e_i \otimes e_{i+n}) a_{i+n} + (-1)^{n+1} a_i f(e_{i+1} \otimes e_{i+n+1}) \\ &= \begin{cases} \lambda_i^{p+1} \bar{a}_{i-1} a_{i-1} - (-1)^n \lambda_{i+1}^{p+1} a_i \bar{a}_i & \text{if } t = m - 1 \\ \mu_i^p a_i a_{i+1} - (-1)^n \mu_{i+1}^p a_i a_{i+1} & \text{if } t = 1 \\ \sigma_i^p a_i + \tau_i^p \bar{a}_{i-1} a_{i-1} a_i - (-1)^n (\sigma_{i+1}^p a_i + \tau_{i+1}^p a_i \bar{a}_i a_i) & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Applying the relations in $\Lambda_{\mathbf{q}}$ gives:

$$d^n(f)(e_i \otimes e_{i+n+1}) = \begin{cases} (q_i \lambda_i^{p+1} - (-1)^n \lambda_{i+1}^{p+1}) a_i \bar{a}_i & \text{if } t = m - 1 \\ 0 & \text{if } t = 1 \\ (\sigma_i^p - (-1)^n \sigma_{i+1}^p) a_i & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $f \in \text{Ker } d^n$ and $t = m - 1$ this gives the condition

$$\begin{aligned}\lambda_{i+1}^{p+1} &= (-1)^n q_i \lambda_i^{p+1} = (-1)^{2n} q_i q_{i-1} \lambda_{i-1}^{p+1} = (-1)^{3n} q_i q_{i-1} q_{i-2} \lambda_{i-2}^{p+1} \\ &= \cdots = (-1)^{mn} q_i q_{i-1} q_{i-2} \cdots q_{i-m+1} \lambda_{i+1}^{p+1}\end{aligned}$$

and hence

$$\lambda_{i+1}^{p+1} = (-1)^{mn} \zeta \lambda_{i+1}^{p+1}.$$

So to get non-trivial solutions for λ_{i+1}^{p+1} we need $\zeta = (-1)^{mn}$. But we assumed that ζ is not a root of unity and thus there are no non-trivial solutions for λ_{i+1}^{p+1} , that is, $\lambda_i^{p+1} = 0$ for all i .

If $f \in \text{Ker } d^n$ and $t = 0$ this gives the condition

$$\sigma_{i+1}^p = (-1)^n \sigma_i^p = (-1)^{2n} \sigma_{i-1}^p = \cdots = (-1)^{mn} \sigma_{i+1}^p$$

and so to get non-trivial solutions for σ_{i+1}^p we need

$$(-1)^{mn} = 1.$$

Now note that each σ_i^p is determined by the others, so we need only determine one of them, say σ_0^p . Then we will have a free choice for σ_0^p if mn is even or $\text{char } K = 2$, but $\sigma_0^p = 0$ (and hence $\sigma_i^p = 0$ for all i) if mn is odd and $\text{char } K \neq 2$.

So if $r = 0$ then, for f to be in $\text{Ker } d^n$, we have the conditions:

$$\begin{cases} \lambda_i^{p+1} = 0 & \text{if } t = m - 1 \\ \sigma_i^p = 0 & \text{if } t = 0 \text{ and } (-1)^{mn} \neq 1 \\ \sigma_i^p = (-1)^{in} \sigma_0^p & \text{if } t = 0 \text{ and } (-1)^{mn} = 1 \end{cases}$$

for all $i = 0, \dots, m - 1$.

We next consider $r = n + 1$. Then

$$\begin{aligned}d^n(f)(e_i \otimes e_{i-n-1}) &= f \circ \partial^{n+1}(e_i \otimes e_{i-n-1}) \\ &= (-1)^{n+1} f(e_i \otimes e_{i-n}) \bar{a}_{i-n-1} + \bar{a}_{i-1} f(e_{i-1} \otimes e_{i-n-1}) \\ &= \begin{cases} -(-1)^n \mu_i^{-p-1} a_i \bar{a}_i + \mu_{i-1}^{-p-1} \bar{a}_{i-1} a_{i-1} & \text{if } t = m - 1 \\ -(-1)^n \lambda_i^{-p} \bar{a}_{i-1} \bar{a}_{i-2} + \lambda_{i-1}^{-p} \bar{a}_{i-1} \bar{a}_{i-2} & \text{if } t = 1 \\ -(-1)^n (\sigma_i^{-p} \bar{a}_{i-1} + \tau_i^{-p} \bar{a}_{i-1} a_{i-1} \bar{a}_{i-1}) + \sigma_{i-1}^{-p} \bar{a}_{i-1} + \tau_{i-1}^{-p} \bar{a}_{i-1} \bar{a}_{i-2} a_{i-2} & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Applying the relations in $\Lambda_{\mathbf{q}}$ gives:

$$d^n(f)(e_i \otimes e_{i-n-1}) = \begin{cases} (q_i \mu_{i-1}^{-p-1} - (-1)^n \mu_i^{-p-1}) a_i \bar{a}_i & \text{if } t = m - 1 \\ 0 & \text{if } t = 1 \\ (\sigma_{i-1}^{-p} - (-1)^n \sigma_i^{-p}) \bar{a}_{i-1} & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $t = 0$ then we will get that the σ_i^{-p} are all dependent on σ_0^{-p} , and they will be all zero if mn is odd and $\text{char } K \neq 2$. If $t = m - 1$ then all the μ_i^{-p-1} are zero.

So if $r = n + 1$ then, for f to be in $\text{Ker } d^n$, we have the conditions:

$$\begin{cases} \mu_i^{-p-1} = 0 & \text{if } t = m - 1 \\ \sigma_i^{-p} = 0 & \text{if } t = 0 \text{ and } (-1)^{mn} \neq 1 \\ \sigma_i^{-p} = (-1)^{in} \sigma_0^{-p} & \text{if } t = 0 \text{ and } (-1)^{mn} = 1. \end{cases}$$

for all $i = 0, \dots, m - 1$.

We now do the generic case for r with $1 \leq r \leq n$. We have

$$\begin{aligned} d^n(f)(e_i \otimes e_{i+n+1-2r}) &= f \circ \partial^{n+1}(e_i \otimes e_{i+n+1-2r}) \\ &= f(e_i \otimes_r e_{i+n-2r}) a_{i+n-2r} - (-1)^n q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} f(e_i \otimes_{r-1} e_{i+n-2r+2}) \bar{a}_{i+n-2r+1} \\ &\quad - (-1)^n ((-1)^r q_{i-r+1} q_{i-r+2} \cdots q_i a_i f(e_{i+1} \otimes_r e_{i+n-2r+1}) + (-1)^r \bar{a}_{i-1} f(e_{i-1} \otimes_{r-1} e_{i+n-2r+1})) \\ &= \begin{cases} \sigma_i^\alpha a_i - (-1)^{n+r} q_{i-r+1} q_{i-r+2} \cdots q_i \sigma_{i+1}^\alpha a_i & \text{if } n - 2r = \alpha m \\ (q_i \lambda_i^\beta - (-1)^n q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} \mu_i^\beta) a_i \bar{a}_i \\ \quad - (-1)^{n+r} (q_{i-r+1} q_{i-r+2} \cdots q_i \lambda_{i+1}^\beta + q_i \mu_{i-1}^\beta) a_i \bar{a}_i & \text{if } n - 2r = \beta m - 1 \\ -(-1)^n q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} \sigma_i^\alpha \bar{a}_{i-1} - (-1)^{n+r} \sigma_{i-1}^\alpha \bar{a}_{i-1} & \text{if } n - 2r = \alpha m - 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $n - 2r = \alpha m$ we get a similar situation to the $r = 0$ and $t = m - 1$ case. We write $r = bm + c$ with $b \in \mathbb{Z}$ and $0 \leq c \leq m - 1$. We need:

$$\begin{aligned} \sigma_i^\alpha &= (-1)^{n+r} \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_i}_{r \text{ terms}} \sigma_{i+1}^\alpha = (-1)^{n+r} \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_{i-r+c}}_{c \text{ terms}} \zeta^b \sigma_{i+1}^\alpha \\ &= (-1)^{2n+2r} \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_{i-r+c}}_{c \text{ terms}} \underbrace{q_{i-r+c+1} q_{i-r+c+2} \cdots q_{i-r+2c}}_{c \text{ terms}} \zeta^{2b} \sigma_{i+2}^\alpha \\ &= \cdots = (-1)^{mn+mr} \zeta^c \zeta^{mb} \sigma_i^\alpha = (-1)^{mn+mr} \zeta^r \sigma_i^\alpha. \end{aligned}$$

Thus either all σ_i^α are zero or ζ is a root of unity. Hence (by assumption) $\sigma_i^\alpha = 0$ for all i and all α with $n - 2r = \alpha m$.

For $n - 2r = \beta m - 1$, the condition that f is in $\text{Ker } d^n$ yields the m equations

$$(-1)^{n+r} \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_i}_{r \text{ terms}} \lambda_{i+1}^\beta = q_i \lambda_i^\beta - (-1)^n \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1}}_{n-r+1 \text{ terms}} \mu_i^\beta - (-1)^{n+r} q_i \mu_{i-1}^\beta$$

in the $2m$ variables $\lambda_i^\beta, \mu_i^\beta$ where $i = 0, \dots, m-1$ (with the obvious conventions that $\lambda_0^\beta = \lambda_{m-1}^\beta$ etc.). We may rewrite these equations as

$$\begin{aligned} \lambda_{i+1}^\beta &= (-1)^{n+r} \underbrace{(q_{i-r+1} q_{i-r+2} \cdots q_{i-1})^{-1}}_{r-1 \text{ terms}} \lambda_i^\beta \\ &\quad + (-1)^{r+1} \underbrace{(q_{i-r+1} q_{i-r+2} \cdots q_i)^{-1}}_{r \text{ terms}} \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1}}_{n-r+1 \text{ terms}} \mu_i^\beta \\ &\quad - \underbrace{(q_{i-r+1} q_{i-r+2} \cdots q_{i-1})^{-1}}_{r-1 \text{ terms}} \mu_{i-1}^\beta \end{aligned}$$

since the q_i are invertible. Thus we may write all the λ_i^β in terms of $\lambda_0^\beta, \mu_0^\beta, \dots, \mu_{m-1}^\beta$. We may then write λ_0^β in terms of $\mu_0^\beta, \dots, \mu_{m-1}^\beta$, provided that the coefficient of λ_0^β is non-zero. Specifically, if $r \neq 1$, then the equations give

$$\begin{aligned} \lambda_0^\beta &= (-1)^{(n+r)m} (q_{m-r} \cdots q_{m-2})^{-1} (q_{m-r-1} \cdots q_{m-3})^{-1} \cdots (q_{m-r+1} \cdots q_{m-1})^{-1} \lambda_0^\beta \\ &\quad + \text{terms in } \mu_0^\beta, \dots, \mu_{m-1}^\beta \end{aligned}$$

and hence

$$\lambda_0^\beta = (-1)^{(n+r)m} (\zeta^{-1})^{r-1} \lambda_0^\beta + \text{terms in } \mu_0^\beta, \dots, \mu_{m-1}^\beta.$$

Since ζ is not a root of unity, it follows that we may write λ_0^β in terms of $\mu_0^\beta, \dots, \mu_{m-1}^\beta$. On the other hand, suppose $r = 1$. Here the original equations reduce to

$$\lambda_{i+1}^\beta = (-1)^{n+1} \lambda_i^\beta + \underbrace{q_{i+1} q_{i+2} \cdots q_{i+n-1}}_{n-1 \text{ terms}} \mu_i^\beta - \mu_{i-1}^\beta.$$

If n is even and $\text{char } K \neq 2$ then we can again write λ_0^β in terms of $\mu_0^\beta, \dots, \mu_{m-1}^\beta$. However, if n is odd or $\text{char } K = 2$ then adding these equations together gives

$$\sum_{i=0}^{m-1} ((q_{i+1} \cdots q_{i+n-1}) - 1) \mu_i^\beta = 0$$

so that there is a dependency among the μ_i^β but λ_0^β is a free variable if $n \neq 1$. (If $n = 1$ then both sides are zero so there is no dependency.)

Finally, we consider the case where $n - 2r = \alpha m - 2$. Here we have the condition:

$$(-1)^n q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r+1} \sigma_i^\alpha = -(-1)^{n+r} \sigma_{i-1}^\alpha.$$

This is similar to the $n - 2r = \alpha m$ case and we deduce that all the σ_i^α are zero since ζ is not a root of unity.

Hence, if $1 \leq r \leq n$ and f is in $\text{Ker } d^n$, we have:

$$\begin{cases} \sigma_i^\alpha = 0 & \text{if } n - 2r = \alpha m \text{ or if } n - 2r = \alpha m - 2 \\ \dim \text{sp}\{\lambda_0^\beta, \dots, \lambda_{m-1}^\beta, \mu_0^\beta, \dots, \mu_{m-1}^\beta\} = m & \text{if } n - 2r = \beta m - 1 \text{ and either } r \neq 1 \text{ or } n \neq 1 \\ \dim \text{sp}\{\lambda_0^\beta, \dots, \lambda_{m-1}^\beta, \mu_0^\beta, \dots, \mu_{m-1}^\beta\} = m + 1 & \text{if } n - 2r = \beta m - 1, r = 1 \text{ and } n = 1. \end{cases}$$

We now combine this information to determine $\dim \text{Ker } d^n$.

Proposition 2.2. *For $m \geq 3$,*

$$\dim \operatorname{Ker} d^n = \begin{cases} m+1 & \text{if } n = 0 \text{ or } n = 1 \\ (2p+1)m & \text{if } n \geq 2. \end{cases}$$

Proof. We first do the cases $n = 0, 1$. If $n = 0$ then $r = 0, 1$ and $\alpha = 0$. Moreover $(-1)^{mn} = 1$, so $\sigma_i^0 = \sigma_0^0$ for all i . Thus $\dim \operatorname{Ker} d^0 = m+1$. If $n = 1$ then we have $r = 0, 1, 2$ and so $n - 2r = 1, -1, -3$ respectively. The only condition comes from the $r = 1$ case, where we have free variables $\lambda_0^0, \mu_0^0, \dots, \mu_{m-1}^0$. Thus $\dim \operatorname{Ker} d^0 = m+1$.

For $n = pm + t \geq 2$ we consider the 4 cases depending on the parity of t and of m .

Suppose both t and m are even. Here we need only consider the possible values of σ_i^α and τ_i^α with $-p \leq \alpha \leq p$. We have that all σ_i^α are zero. (Note that if $t = 0$ so $n = pm$ then the $r = 1$ case where $n - 2 = pm - 2$ shows that all the σ_i^p are zero and the $r = n$ case where $n - 2n = -pm$ shows that all the σ_i^{-p} are zero.) Hence the only contribution to the kernel is from the τ_i^α and thus $\dim \operatorname{Ker} d^n = (2p+1)m$.

Suppose t even and m odd. If $t \neq m-1$, we have $(p+1)m$ many σ_i^α , $(p+1)m$ many τ_i^α , pm many λ_i^β and pm many μ_i^γ . All the σ_i^α are zero and the τ_i^α are free as for the previous case giving a $(p+1)m$ dimensional contribution. The dependence between the λ_i^β and μ_i^β gives another pm dimensional contribution. Thus $\dim \operatorname{Ker} d^n = (2p+1)m$ if $t \neq m-1$. The case $t = m-1$ is similar, and we note that λ_i^{p+1} and μ_i^{-p-1} are all zero by the $r = 0$ and $r = n+1$ cases respectively. So $\dim \operatorname{Ker} d^n = (2p+1)m$ if $t = m-1$.

Suppose t odd and m even. If $t \leq m-2$ we have $(2p+1)m$ many λ_i^β and $(2p+1)m$ many μ_i^γ . Thus the dependence between the λ_i^β and μ_i^β gives a $(2p+1)m$ dimensional contribution. If $t = m-1$ then we have $\lambda_i^{p+1} = 0$ and $\mu_i^{-p-1} = 0$ from the $r = 0$ and $r = n+1$ cases so we still get $(2p+1)m$ dimensions. Thus $\dim \operatorname{Ker} d^n = (2p+1)m$.

Finally we consider the case where t and m are both odd. We have pm values of σ_i^α , pm values of τ_i^α , $(p+1)m$ values of λ_i^β and $(p+1)m$ values of μ_i^γ . Again, the dependence between the λ_i^β and μ_i^β gives a $(p+1)m$ dimensional contribution. The σ_i^α are all zero and the τ_i^α are free. Hence $\dim \operatorname{Ker} d^n = (2p+1)m$.

This completes the proof. \square

Using the rank-nullity theorem we now get the dimension of $\operatorname{Im} d^{n-1}$.

Proposition 2.3. *For $m \geq 3$ and $n = pm + t$ we have*

$$\dim \operatorname{Im} d^{n-1} = \begin{cases} 0 & \text{if } n = 0 \\ m-1 & \text{if } n = 1 \text{ or } n = 2 \\ (2p+1)m & \text{if } n \geq 3. \end{cases}$$

Proof. The cases $n = 0, 1, 2$ are immediate. For $n \geq 3$, write $n = pm + t$ with $0 \leq t \leq m-1$. If $t \neq 0$ then $\dim_K \operatorname{Hom}(P^{pm+t-1}, \Lambda_{\mathbf{q}}) = (4p+2)m$ and $\dim \operatorname{Ker} d^{pm+t-1} = (2p+1)m$. If $t = 0$

then $\dim_K \operatorname{Hom}(P^{pm-1}, \Lambda_{\mathbf{q}}) = 4pm$ and $\dim \operatorname{Ker} d^{pm-1} = (2p-1)m$. Thus in both cases we have $\dim \operatorname{Im} d^{n-1} = (2p+1)m$. \square

We come now to our main results where we determine the Hochschild cohomology ring of the algebra $\Lambda_{\mathbf{q}}$ when ζ is not a root of unity.

thm:dimHH

Theorem 2.4. *For $m \geq 3$,*

$$\dim \operatorname{HH}^n(\Lambda_{\mathbf{q}}) = \begin{cases} m+1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3. \end{cases}$$

Thus $\operatorname{HH}^(\Lambda_{\mathbf{q}})$ is a finite-dimensional algebra of dimension $m+4$.*

thm:ring

Theorem 2.5. *For $m \geq 3$, we have*

$$\operatorname{HH}^*(\Lambda_{\mathbf{q}}) \cong K[x_0, x_1, \dots, x_{m-1}]/(x_i x_j) \times_K \bigwedge(u_1, u_2)$$

where \times_K denotes the fibre product over K , $\bigwedge(u_1, u_2)$ is the exterior algebra on the generators u_1 and u_2 , the x_i are in degree 0, and the u_i are in degree 1.

Proof. Since $\operatorname{HH}^0(\Lambda_{\mathbf{q}})$ is the centre $Z(\Lambda_{\mathbf{q}})$, it is clear that $\operatorname{HH}^0(\Lambda_{\mathbf{q}})$ has K -basis $\{1, x_0, \dots, x_{m-1}\}$ where $x_i = a_i \bar{a}_i$. Thus $\operatorname{HH}^0(\Lambda_{\mathbf{q}}) = K[x_0, x_1, \dots, x_{m-1}]/(x_i x_j)$.

Define bimodule maps $u_1, u_2 : P^1 \rightarrow \Lambda_{\mathbf{q}}$ by

$$u_1 : \begin{cases} \mathfrak{o}(g_{0,i}^1) \otimes \mathfrak{t}(g_{0,i}^1) & \mapsto a_i \text{ for all } i = 0, 1, \dots, m-1 \\ \text{else} & \mapsto 0, \end{cases}$$

$$u_2 : \begin{cases} \mathfrak{o}(g_{0,m-1}^1) \otimes \mathfrak{t}(g_{0,m-1}^1) & \mapsto a_{m-1} \\ \mathfrak{o}(g_{1,0}^1) \otimes \mathfrak{t}(g_{1,0}^1) & \mapsto \bar{a}_{m-1} \\ \text{else} & \mapsto 0. \end{cases}$$

It is straightforward to show that these maps are in $\operatorname{Ker} d^1$ and that they represent linearly independent elements in $\operatorname{HH}^1(\Lambda_{\mathbf{q}})$ which we also denote by u_1 and u_2 . Hence $\{u_1, u_2\}$ is a K -basis for $\operatorname{HH}^1(\Lambda_{\mathbf{q}})$.

In order to show that $u_1 u_2$ represents a non-zero element of $\operatorname{HH}^2(\Lambda_{\mathbf{q}})$, we define bimodule maps $\mathcal{L}^0(u_2) : P^1 \rightarrow P^0$ and $\mathcal{L}^1(u_2) : P^2 \rightarrow P^1$ by

$$\mathcal{L}^0(u_2) : \begin{cases} \mathfrak{o}(g_{0,m-1}^1) \otimes \mathfrak{t}(g_{0,m-1}^1) & \mapsto a_{m-1} \otimes e_0 \\ \mathfrak{o}(g_{1,0}^1) \otimes \mathfrak{t}(g_{1,0}^1) & \mapsto \bar{a}_{m-1} \otimes e_{m-1} \\ \text{else} & \mapsto 0, \end{cases}$$

$$\mathcal{L}^1(u_2) : \begin{cases} \mathfrak{o}(g_{0,m-1}^2) \otimes \mathfrak{t}(g_{0,m-1}^2) & \mapsto a_{m-1} \mathfrak{o}(g_{0,0}^1) \otimes \mathfrak{t}(g_{0,0}^1) \\ \mathfrak{o}(g_{1,0}^2) \otimes \mathfrak{t}(g_{1,0}^2) & \mapsto \bar{a}_{m-1} \mathfrak{o}(g_{0,m-1}^1) \otimes \mathfrak{t}(g_{0,m-1}^1) \\ \mathfrak{o}(g_{1,m-1}^2) \otimes \mathfrak{t}(g_{1,m-1}^2) & \mapsto -q_{m-1} a_{m-1} \mathfrak{o}(g_{1,0}^1) \otimes \mathfrak{t}(g_{1,0}^1) \\ \mathfrak{o}(g_{2,0}^2) \otimes \mathfrak{t}(g_{2,0}^2) & \mapsto -\bar{a}_{m-1} \mathfrak{o}(g_{1,m-1}^1) \otimes \mathfrak{t}(g_{1,m-1}^1) \\ \text{else} & \mapsto 0. \end{cases}$$

Then the following diagram is commutative

$$\begin{array}{ccccc}
 P^2 & \xrightarrow{\partial^2} & P^1 & & \\
 \mathcal{L}^1(u_2) \downarrow & & \mathcal{L}^0(u_2) \downarrow & \searrow u_2 & \\
 P^1 & \xrightarrow{\partial^1} & P^0 & \longrightarrow & \Lambda_{\mathbf{q}}
 \end{array}$$

where $P^0 \rightarrow \Lambda_{\mathbf{q}}$ is the multiplication map. Thus the element $u_1 u_2 \in \mathrm{HH}^2(\Lambda_{\mathbf{q}})$ is represented by the map $u_1 \circ \mathcal{L}^1(u_2) : P^2 \rightarrow \Lambda_{\mathbf{q}}$, that is, by the map

$$\begin{cases} \mathfrak{o}(g_{1,0}^2) \otimes \mathfrak{t}(g_{1,0}^2) & \mapsto \bar{a}_{m-1} a_{m-1} \\ \text{else} & \mapsto 0. \end{cases}$$

Since this map is not in $\mathrm{Im} d^1$, it follows that $u_1 u_2$ is non-zero in $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$ and hence $\mathrm{HH}^2(\Lambda_{\mathbf{q}}) = \mathrm{sp}\{u_1 u_2\}$.

From the lifting $\mathcal{L}^1(u_2)$ it is easy to see that u_2^2 represents the zero element in $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$, and a similar calculation shows that u_1^2 also represents the zero element in $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$. (Note that although it is immediate from the graded commutativity of $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ that $u_1^2 = 0 = u_2^2$ in $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$ when $\mathrm{char} K \neq 2$, this direct calculation is required when $\mathrm{char} K = 2$.)

Thus we have elements u_1 and u_2 in $\mathrm{HH}^1(\Lambda_{\mathbf{q}})$ which are annihilated by all the $x_i \in \mathrm{HH}^0(\Lambda_{\mathbf{q}})$ and with $u_1^2 = 0 = u_2^2$ and $u_1 u_2 = -u_2 u_1$ (with the latter by the graded-commutativity of $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$). Thus

$$\mathrm{HH}^*(\Lambda_{\mathbf{q}}) \cong K[x_0, x_1, \dots, x_{m-1}]/(x_i x_j) \times_K \bigwedge(u_1, u_2)$$

where \times_K denotes the fibre product over K , $\bigwedge(u_1, u_2)$ is the exterior algebra on the generators u_1 and u_2 , the x_i are in degree 0, and the u_i are in degree 1. \square

3. THE CASE $m = 2$.

$m=2$

We assume that $m = 2$ throughout this section. Recall from Lemma 2.1 that $\dim_K \mathrm{Hom}(P^n, \Lambda_{\mathbf{q}}) = 4(n+1)$. For $f \in \mathrm{Hom}(P^n, \Lambda_{\mathbf{q}})$ we may write:

$$\begin{cases} f(e_i \otimes e_{i+2\alpha}) = \sigma_i^\alpha e_i + \tau_i^\alpha \bar{a}_{i-1} a_{i-1} & \text{if } n \text{ even} \\ f(e_i \otimes e_{i+2\beta+1}) = \lambda_i^\beta \bar{a}_{i-1} + \mu_i^\beta a_i & \text{if } n \text{ odd,} \end{cases}$$

with coefficients $\sigma_i^\alpha, \tau_i^\alpha, \lambda_i^\beta$ and μ_i^β in K . The choices of α and β are:

$$\begin{cases} -p \leq \alpha \leq p & \text{if } n \text{ is even} \\ -p-1 \leq \beta \leq p & \text{if } n \text{ is odd,} \end{cases}$$

which gives $n+1$ values in each case.

3.1. **Ker d^n where $m = 2$.** Let $f \in \text{Hom}(P^n, \Lambda_{\mathbf{q}})$ and suppose $f \in \text{Ker } d^n$ so that $d^n(f) = f \circ \partial^{n+1} \in \text{Hom}(P^{n+1}, \Lambda_{\mathbf{q}})$. Write $n = 2p + t$ with $t = 0, 1$. We evaluate $d^n(f)$ at $e_i \otimes e_{i+n+1-2r}$ for $r = 0, \dots, n+1$. There are three cases to consider.

We first consider $r = 0$. Then, for $i = 0, 1$ and after applying the relations in $\Lambda_{\mathbf{q}}$ we have:

$$d^n(f)(e_i \otimes e_{i+n+1}) = \begin{cases} (q_i \lambda_i^p + \lambda_{i+1}^p) a_i \bar{a}_i & \text{if } t = 1 \\ (\sigma_i^p - \sigma_{i+1}^p) a_i & \text{if } t = 0. \end{cases}$$

Thus if $f \in \text{Ker } d^n$ and $t = 1$ this gives the condition

$$\lambda_{i+1}^p = -q_i \lambda_i^p = q_i q_{i-1} \lambda_{i+1}^p = \zeta \lambda_{i+1}^p$$

and hence $\lambda_1^p = \lambda_0^p = 0$ as $\zeta \neq 1$. If $t = 0$ this gives the condition $\sigma_{i+1}^p = \sigma_i^p$ and so there is a one-parameter solution here.

We next consider $r = n+1$. After applying the relations in $\Lambda_{\mathbf{q}}$ we have:

$$d^n(f)(e_i \otimes e_{i-n-1}) = \begin{cases} (q_i \mu_{i-1}^{-p-1} + \mu_i^{-p-1}) a_i \bar{a}_i & \text{if } t = 1 \\ (\sigma_{i-1}^{-p} - \sigma_i^{-p}) \bar{a}_{i-1} & \text{if } t = 0. \end{cases}$$

If $t = 0$ then we get a one-parameter solution for the σ_i^{-p} as before. If $t = 1$ then we have $\mu_0^{-p-1} = \mu_1^{-p-1} = 0$.

We now do the generic case for r with $1 \leq r \leq n$. We have $d^n(f)(e_i \otimes e_{i+n+1-2r}) =$

$$\begin{cases} (q_i \lambda_i^{p-2\epsilon} + \zeta^{p-\epsilon+1} \mu_i^{p-2\epsilon+1} + \zeta^\epsilon \lambda_{i+1}^{p-2\epsilon} + q_i \mu_{i+1}^{p-2\epsilon+1}) a_i \bar{a}_i & \text{if } n \text{ odd and } r = 2\epsilon \\ q_i (\lambda_i^{p-2\epsilon-1} + \zeta^{p-\epsilon} \mu_i^{p-2\epsilon} - \zeta^\epsilon \lambda_{i+1}^{p-2\epsilon-1} - \mu_{i+1}^{p-2\epsilon}) a_i \bar{a}_i & \text{if } n \text{ odd and } r = 2\epsilon + 1 \\ (\sigma_i^{p-2\epsilon} - \zeta^\epsilon \sigma_{i+1}^{p-2\epsilon}) a_i - (q_{i+1} \zeta^{p-\epsilon} \sigma_i^{p-2\epsilon+1} - \sigma_{i+1}^{p-2\epsilon+1}) \bar{a}_{i+1} & \text{if } n \text{ even and } r = 2\epsilon \\ (\sigma_i^{p-2\epsilon-1} - q_i \zeta^\epsilon \sigma_{i+1}^{p-2\epsilon-1}) a_i + (-\zeta^{p-\epsilon} \sigma_i^{p-2\epsilon} + \sigma_{i+1}^{p-2\epsilon}) \bar{a}_{i+1} & \text{if } n \text{ even and } r = 2\epsilon + 1. \end{cases}$$

If n is even, then all the σ_i^α 's are zero and there is no condition on the τ_i^α 's, as for the $m \geq 3$ case.

Now suppose n is odd. If we fix n odd and r even with $1 \leq r \leq n$ (so that $n \geq 3$) we get a pair of equations:

$$\begin{aligned} q_i \lambda_i^{p-2\epsilon} + \zeta^{p-\epsilon+1} \mu_i^{p-2\epsilon+1} + \zeta^\epsilon \lambda_{i+1}^{p-2\epsilon} + q_i \mu_{i+1}^{p-2\epsilon+1} &= 0 \\ \zeta^\epsilon \lambda_i^{p-2\epsilon} + q_{i+1} \mu_i^{p-2\epsilon+1} + q_{i+1} \lambda_{i+1}^{p-2\epsilon} + \zeta^{p-\epsilon-1} \mu_{i+1}^{p-2\epsilon+1} &= 0. \end{aligned}$$

These equations have a two-parameter solution if and only if $q_i q_{i+1} - \zeta^{p+1} = \zeta(1 - \zeta^p) \neq 0$. Since $p \geq 1$, this is non-zero, and we have a two-parameter solution.

If we fix n odd and r odd with $1 \leq r \leq n$ we get a pair of equations:

$$\begin{aligned} \lambda_i^{p-2\epsilon-1} + \zeta^{p-\epsilon} \mu_i^{p-2\epsilon} - \zeta^\epsilon \lambda_{i+1}^{p-2\epsilon-1} - \mu_{i+1}^{p-2\epsilon} &= 0 \\ -\zeta^\epsilon \lambda_i^{p-2\epsilon-1} - \mu_i^{p-2\epsilon} + \lambda_{i+1}^{p-2\epsilon-1} + \zeta^{p-\epsilon} \mu_{i+1}^{p-2\epsilon} &= 0. \end{aligned}$$

These equations have a two-parameter solution if and only if $-1 + \zeta^p \neq 0$. For $n \geq 3$ we have $p \geq 1$, so this is non-zero, and we have a two-parameter solution. If however $n = 1$ then necessarily

$r = n = 1$, and we get

$$\lambda_i^{-1} + \mu_i^0 - \lambda_{i+1}^{-1} - \mu_{i+1}^0 = 0$$

which has a three-parameter solution.

We may now determine the dimension of $\text{Ker } d^n$ for $m = 2$.

Proposition 3.1. *For $m = 2$ and $n = 2p + t$ with $t = 0, 1$, we have*

$$\dim \text{Ker } d^n = \begin{cases} 3 & \text{if } n = 0 \text{ or } n = 1 \\ 2(2p + 1) & \text{if } n \geq 2, \end{cases}$$

and

$$\dim \text{Im } d^n = \begin{cases} 1 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 2(2p + 3) & \text{if } n \geq 2 \text{ and } n \text{ odd} \\ 2(2p + 1) & \text{if } n \geq 2 \text{ and } n \text{ even.} \end{cases}$$

Proof. We first do the cases with small values of n . If $n = 0$ then $r = 0, 1$ and $\alpha = 0$. Thus we get $\dim \text{Ker } d^0 = 3$, corresponding to the free variables σ_0^0 , τ_0^0 and τ_1^0 , and hence the image is one-dimensional. If $n = 1$, we have $r = 0, 1, 2$ and $n - 2r = 1, -1, -3$ respectively. We need to determine μ_i^0 , μ_i^{-1} , λ_i^0 and λ_i^{-1} (8 variables in total). The $r = 0$ case gives $\lambda_i^0 = 0$, and the $r = 2$ case gives $\mu_i^{-1} = 0$. The $n = r = 1$ case has a three-parameter solution for λ_i^{-1} in terms of μ_i^0 , μ_{i+1}^0 and λ_{i+1}^{-1} . Thus overall we get $\dim \text{Ker } d^1 = 3$ and $\dim \text{Im } d^1 = 5$.

We next consider the case $n = 2p$; we need only consider the possible values of σ_i^α and τ_i^α with $-p \leq \alpha \leq p$. Here we get that all σ_i^α are zero. So the only contribution to the kernel is from the τ_i^α and thus the kernel has dimension $2(2p + 1)$. The dimension of the image is thus $4(2p + 1) - 2(2p + 1) = 2(2p + 1)$.

Finally, we consider $n = 2p + 1$. Here we have $2(2p + 2)$ many λ_i^β and $2(2p + 2)$ many μ_i^β with $-p - 1 \leq \beta \leq p$. All the λ_i^β are dependent on the $\mu_i^{\beta+1}$ if $-p - 1 \leq \beta \leq p - 1$, so we get a $2(2p + 1)$ dimensional contribution here. Moreover $\lambda_0^p = 0 = \lambda_1^p$ and $\mu_0^{-p-1} = 0 = \mu_1^{-p-1}$. Hence $\dim \text{Ker } d^n = 2(2p + 1)$. This gives $\dim \text{Im } d^n = 4(2p + 2) - 2(2p + 1) = 2(2p + 3)$. \square

Noting that $\dim \text{HH}^0(\Lambda_{\mathbf{q}}) = 3 = m + 1$, we combine these results with Theorem 2.4 to give the following theorem.

Theorem 3.2. *For $m \geq 2$,*

$$\dim \text{HH}^n(\Lambda_{\mathbf{q}}) = \begin{cases} m + 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3. \end{cases}$$

Thus $\text{HH}^*(\Lambda_{\mathbf{q}})$ is a finite-dimensional algebra of dimension $m + 4$.

It can be verified directly that the proof of Theorem 2.5 also holds when $m = 2$. Hence we have the following result which describes the ring structure of $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ when $m = 2$ and ζ is not a root of unity.

Theorem 3.3. *For $m = 2$, we have*

$$\mathrm{HH}^*(\Lambda_{\mathbf{q}}) \cong K[x_0, x_1]/(x_i x_j) \times_K \bigwedge(u_1, u_2)$$

where \times_K denotes the fibre product over K , $\bigwedge(u_1, u_2)$ is the exterior algebra on the generators u_1 and u_2 , and the elements x_0, x_1 are in degree 0 and u_1, u_2 in degree 1.

We end by remarking that we have exhibited self-injective algebras whose Hochschild cohomology ring is of arbitrarily large, but nevertheless finite, dimension. The case $m = 1$ was studied in [2] where it was shown that the Hochschild cohomology ring is 5-dimensional when ζ is not a root of unity. Thus, for all $m \geq 1$, we now have self-injective algebras whose Hochschild cohomology ring is $(m + 4)$ -dimensional. Hence, for each $N \geq 5$ we have an algebra with $N - 4$ simple modules, of dimension $4(N - 4)$ and with infinite global dimension whose Hochschild cohomology ring is N -dimensional.

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